L^2-INVISIBILITY AND A CLASS OF LOCAL SIMILARITY GROUPS

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ABSTRACT. In this note we show that the members of a certain class of local similarity groups are l^2-invisible, i.e. the (non-reduced) group homology of the regular unitary representation vanishes in all degrees. This class contains groups of type FP_∞, e.g. Thompson's group V and Nekrashevych-Röver groups. They yield counterexamples to a generalized zero-in-the-spectrum conjecture for groups of type FP_∞.

1. INTRODUCTION

The *zero-in-the-spectrum conjecture* (or question) appears for the first time in Gromov's article [7]. It states that for an aspherical closed Riemannian manifold M there always exists $p \geq 0$ such that the spectrum of the Laplacian Δ_p acting on the square integrable p-forms on the universal covering of M contains zero. The latter is equivalent (see [10]) to the group-homological statement

$$\exists_{p\geq 0}\ H_p\big(\Gamma, \mathcal{N}(\Gamma)\big) \neq 0, \tag{1}$$

where $\Gamma = \pi_1(M)$ and $\mathcal{N}(\Gamma)$ is the group von Neumann algebra of Γ. The zero-in-the-spectrum conjecture is motivated and implied by the strong Novikov conjecture [7, 4.B.; 12, Theorem 12.7 on p. 443]. We call a group Γ *l^2-invisible* if

$$\forall_{p\geq 0}\ H_p\big(\Gamma, \mathcal{N}(\Gamma)\big) = 0.$$

By [12, Lemmas 6.98 on p. 286 and 12.3 on p. 438] a group Γ of type FP_∞ is l^2-invisible iff

$$\forall_{p\geq 0}\ H_p\big(\Gamma, l^2(\Gamma)\big) = 0.$$

Note that l^2-invisibility of a group is a much stronger property than the vanishing of its l^2-Betti numbers, which is equivalent to the vanishing of the reduced homology.

A more general zero-in-the-spectrum question by Lott [10], where one drops the asphericity condition, was answered in the negative by Farber and Weinberger [5]. This note is concerned with an algebraic generalization of the zero-in-the-spectrum question, which was raised – in different terminology – by Lück [11, Remark 12.16; 12, Remark 12.4 on p. 440]:

(2) *Are there groups of type FP or FP_∞ that are l^2-invisible?*

Here type FP means type FP_∞ together with finite cohomological dimension. Without any finiteness condition on the group, l^2-invisible groups are easily constructed by taking suitable infinite products (see *loc. cit.*). In the spirit of the zero-in-the-spectrum conjecture one might expect the answer to (2) to be negative, but, in fact, we provide here many examples of FP_∞-groups that are l^2-invisible.

Hughes [8] introduced a certain class of groups acting on compact ultrametric spaces which we call *local similarity groups* for short (see Section 2 for details). Assuming there are only finitely many Sim-equivalence classes of balls and the

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similarity structure satisfies a condition called *rich in ball contractions*, these groups satisfy property FP_∞ [6]. In Section 3 we will introduce another property for similarity structures, called *dually contracting*, which is implied by rich in ball contractions and enables us to prove the following theorem in Section 4.

Theorem 1.1. *Let X be a compact ultrametric space with similarity structure* Sim*. If* Sim *is dually contracting, then the local similarity group $\Gamma = \Gamma(\mathrm{Sim})$ is l^2-invisible.*

The well known Thompson group V can be realized as a local similarity group which is contained in this class, as well as the Nekrashevych-Röver groups $V_d(H)$ (Example 2.2). They are also of type FP_∞ by the results in [6]. Already Brown showed in [3, Theorem 4.17] that V is of type FP_∞. Unfortunately, we cannot say anything about the FP-part of question (2) since the groups we consider here are easily seen to have infinite cohomological dimension. Indeed, as a byproduct of our argument, we obtain the following statement which implies infinite cohomological dimension [2, Prop. (6.1) on p. 199 and (6.7) on p. 202].

Theorem 1.2. *Let X be a compact ultrametric space with similarity structure* Sim*. If there are only finitely many* Sim*-equivalence classes of balls and* Sim *is rich in ball contractions, then the local similarity group $\Gamma = \Gamma(\mathrm{Sim})$ satisfies $H^*(\Gamma, \mathbb{Z}[\Gamma]) = 0$ in all degrees.*

Note that the case $\Gamma = V$ has already been treated in [3, Theorem 4.21].

Related work: In [15] Oguni defines an algorithm which takes a finitely presented non-amenable group G as input and gives a finitely presented group G_Ψ with $H_p(G_\Psi, \mathcal{N}(G_\Psi)) = 0$ for all p. But it is not known when G_Ψ is of type FP_∞. In fact, G_Ψ is not even FP_3 if G is a free group.

2. Local similarity groups

In this section we review the definition and fix the terminology for Hughes' class of local similarity groups.

Recall that an *ultrametric space* is a metric space (X, d) such that

$$d(x, y) \leq \max\{d(x, z), d(z, y)\} \quad \text{for all } x, y, z \in X.$$

In this paper, *X always denotes a compact ultrametric space.* The endspace of a rooted proper $\mathbb{R}$-tree is a compact ultrametric space and, conversely, every compact ultrametric space of diameter less than or equal to one is the endspace of a rooted proper $\mathbb{R}$-tree. See [9] for more information in this direction. By a ball in X, we always mean a subset of the form

$$B(x, r) = \{y \in X \mid d(x, y) \leq r\}$$

with $x \in X$ and $r \geq 0$. Two balls are always either disjoint or one contains the other. A non-empty subset is open and closed if and only if it is a union of finitely many balls. Let X, Y be compact ultrametric spaces. A homeomorphism $\gamma : X \to Y$ is called

- an *isometry* iff $d(\gamma(x_1), \gamma(x_2)) = d(x_1, x_2)$ for all $x_1, x_2 \in X$.
- a *similarity* iff there is a $\lambda > 0$ with $d(\gamma(x_1), \gamma(x_2)) = \lambda d(x_1, x_2)$ for all $x_1, x_2 \in X$.
- a *local similarity* iff for every $x \in X$ there are balls $A \subset X$ and $B \subset Y$ with $x \in A$, $\gamma(x) \in B$ and $\gamma|_A : A \to B$ is a similarity.

The set of all local similarities on X forms a group and is denoted by $LS(X)$.

Definition 2.1 ([8, Definition 3.1])**.** Let X be a compact ultrametric space. A *similarity structure* Sim on X (called *finite similarity structure* in [6, 8]) consists of a finite set $\mathrm{Sim}(B_1, B_2)$ of similarities $B_1 \to B_2$ for every ordered pair of balls (B_1, B_2) such that the following axioms are satisfied:

- (Identities) Each $\mathrm{Sim}(B, B)$ contains the identity.
- (Inverses) If $\gamma \in \mathrm{Sim}(B_1, B_2)$ then also $\gamma^{-1} \in \mathrm{Sim}(B_2, B_1)$.
- (Compositions) If $\gamma_1 \in \mathrm{Sim}(B_1, B_2)$ and $\gamma_2 \in \mathrm{Sim}(B_2, B_3)$ then also $\gamma_2 \circ \gamma_1 \in \mathrm{Sim}(B_1, B_3)$.
- (Restrictions) If $\gamma \in \mathrm{Sim}(B_1, B_2)$ and $B_3 \subset B_1$ is a subball then also $\gamma|_{B_3} \in \mathrm{Sim}(B_3, \gamma(B_3))$.

A local similarity $\gamma : X \to X$ is locally determined by Sim iff for every $x \in X$ there is a ball $x \in B \subset X$ such that $\gamma(B)$ is a ball and $\gamma|_B \in \mathrm{Sim}(B, \gamma(B))$. The set of all local similarities locally determined by Sim forms a group, denoted by $\Gamma(\mathrm{Sim})$, and is called the *local similarity group associated to* (X, Sim). A group arising this way is called a *local similarity group*.

Example 2.2 (cf. [8, Section 4])**.** We recall the alphabet terminology of the rooted d-ary tree. Let $A = \{a_1, ..., a_d\}$ be a set of d letters. A word in A is just an element of A^n for some $n \geq 1$ or the empty word. An infinite word is an element in the countable product $A^\omega = \prod_{\mathbb{N}} A$. The simplicial tree associated to A has words as vertices and an edge between to words v, w iff there is an $x \in A$ with $vx = w$ or $v = wx$. The root is the empty word. The endspace of this tree can be identified with the set of infinite words. It comes with a natural ultrametric defined by

$$d(x, y) := \begin{cases} 0 & \text{if } x = y \\ \exp(1 - n) & \text{if } n = \min\{k \mid x_k \neq y_k\} \end{cases}$$

where $x = x_1 x_2 ...$ and $y = y_1 y_2 ...$ are infinite words. Since the tree is locally finite, the endspace with this metric is compact. Call it X.
Now let H be a subgroup of the symmetric group Σ_d of A. Define a similarity structure Sim on X as follows. If B_1 and B_2 are balls of X then there are unique words w_1 and w_2 such that $B_1 = w_1 A^\omega$ and $B_2 = w_2 A^\omega$. If $\sigma \in H$ then

$$\gamma_\sigma : w_1 A^\omega \to w_2 A^\omega \qquad w_1 x_1 x_2 ... \mapsto w_2 \sigma(x_1) \sigma(x_2) ...$$

defines a similarity $B_1 \to B_2$. Set $\mathrm{Sim}(B_1, B_2) := \{\gamma_\sigma \mid \sigma \in H\}$. This defines a similarity structure Sim on X. The corresponding local similarity group is the Nekrashevych-Röver group $V_d(H)$ considered in [14] and [16]. In the case $H = 1$, this specializes to the Higman-Thompson groups V_d and in particular to the well known Thompson group V for $d = 1$.

If A and B are balls in X, we say that A and B are Sim-*equivalent* iff there exists a similarity $A \to B$ in Sim. Denote by $[A]$ the corresponding equivalence class of A. More generally, if Y and Z are non-empty closed open subspaces of X, we say that Y and Z are *locally* Sim-*equivalent* iff there exists a local similarity $Y \to Z$ locally determined by Sim. Denote by $\langle Y \rangle$ the corresponding equivalence class of Y. Of course, two balls are locally Sim-equivalent if they are Sim-equivalent.

If $Y \subset X$ is a non-empty closed open subspace, then one can restrict the similarity structure Sim to one on Y by defining

$$\mathrm{Sim}|_Y := \{\gamma \in \mathrm{Sim} \mid \mathrm{dom}(\gamma) \cup \mathrm{codom}(\gamma) \subset Y\} \cup \{\mathrm{id}_B \mid B \text{ a ball of } Y\}$$

where $\mathrm{dom}(\gamma)$ is the domain of γ and $\mathrm{codom}(\gamma)$ is the codomain of γ. The group $\Gamma(\mathrm{Sim}|_Y)$ is a subgroup of $\Gamma(\mathrm{Sim})$. More precisely, $\Gamma(\mathrm{Sim}|_Y)$ is isomorphic to the subgroup of $\Gamma(\mathrm{Sim})$ consisting of the elements $\alpha : X \to X$ with $\alpha(x) = x$ for $x \in X \setminus Y$. The proof of the next lemma is easy and left to the reader.

Lemma 2.3. *Let X be a compact ultrametric space and* Sim *a similarity structure on X. Let $Y, Z \subset X$ be two non-empty closed open subspaces with $\langle Y\rangle = \langle Z\rangle$. Then the groups $\Gamma(\mathrm{Sim}|_Y)$ and $\Gamma(\mathrm{Sim}|_Z)$ are isomorphic.*

Let X be a compact ultrametric space. There is a rooted locally finite simplicial tree associated to X, called the ball hierarchy. It has balls of X as vertices and an edge between the balls A and B whenever A is a proper maximal subball of B or vice versa. Take the ball X as root. It is locally finite because X is compact. Now define the *depth* of a ball B in X, denoted by $\mathrm{depth}(B)$, to be the distance between the vertex B and the root X in the ball hierarchy tree. We will need the following lemma in Section 3.

Lemma 2.4. *Let X be a compact ultrametric space and $\mathcal{P}$ a partition of X into non-empty closed open subspaces, i.e. $\mathcal{P}$ is a finite set of pairwise disjoint non-empty closed open subspaces of X so that the union of the elements of $\mathcal{P}$ is all of X. Then there exists $N \in \mathbb{N}$ such that every ball with depth at least N is contained in some $P \in \mathcal{P}$.*

Proof. Since every non-empty closed open subspace in a compact ultrametric space is a finite union of balls, we can assume without loss of generality that each $P \in \mathcal{P}$ is a ball. Set

$$N := \max\{\mathrm{depth}(P) \mid P \in \mathcal{P}\}.$$

We claim that every ball with depth at least N is contained in some $P \in \mathcal{P}$. Assume the contradiction. Then there exists a ball B such that $\mathrm{depth}(B) \geq \mathrm{depth}(P)$ for all $P \in \mathcal{P}$ but $B \not\subset P$ for all $P \in \mathcal{P}$. The latter means that for each $P \in \mathcal{P}$ either $B \cap P = \emptyset$ or $P \subsetneq B$. But $P \in \mathcal{P}$ cannot be a proper subball of B because of the depth condition. So we have $B \cap P = \emptyset$ for all $P \in \mathcal{P}$ which contradicts $X = \bigcup_{P \in \mathcal{P}} P$. □

Definition 2.5. We call $\gamma : A \to B$ in a similarity structure Sim

- *contracting* iff $A \subsetneq B$ or $B \subsetneq A$.
- *separating* iff $A \cap B = \emptyset$.
- *equalizing* iff $A = B$.

In general, the precise relationship between the similarity structure and the corresponding local similarity group is not yet understood very well. The following two propositions are easy results in this direction.

Proposition 2.6. *Let X be a compact ultrametric space and* Sim *a similarity structure on X. Then the following are equivalent.*

i) $\Gamma(\mathrm{Sim})$ *is finite.*
ii) *There are only finitely many separating elements in* Sim.
iii) *There are only finitely many non-identity elements in* Sim.

In this case, Sim *contains no contracting elements and* $\Gamma(\mathrm{Sim})$ *fixes all points of X except a finite subset of isolated points. It permutes these isolated points in a way such that* $\Gamma(\mathrm{Sim}) \cong \Sigma_{d_1} \times \ldots \times \Sigma_{d_n}$ *is a finite product of finite symmetric groups.*

Proof. First we make a series of observations.

Observation 1: If $\gamma : A \to B$ is a separating element in Sim, then we can construct an element $\alpha \in \Gamma(\mathrm{Sim})$ by defining $\alpha|_A = \gamma$ and $\alpha|_B = \gamma^{-1}$ and $\alpha(x) = x$ for all other elements $x \in X$. If $\gamma_i : A_i \to B_i$ with $i = 1, 2$ are two separating elements and $\gamma_1 \neq \gamma_2$, then the corresponding α_i also satisfy $\alpha_1 \neq \alpha_2$. In particular, if there are infinitely many distinct such γ_i, then $\Gamma(\mathrm{Sim})$ is infinite.

Observation 2: Assume there is a contracting element $\gamma : A \to B$ in Sim. Assume without loss of generality $B \subsetneq A$. Then there are infinitely many distinct separating

elements in Sim which can be constructed as follows. Let C be a ball in $A \setminus B$ and define $C_i = \gamma^i(C)$ for $i \in \{0, 1, 2, ...\}$. Fix some i. Observe $\gamma^{i+k}(C) \subset \gamma^i(B)$ for all $k \geq 1$ and $\gamma^i(C) \cap \gamma^i(B) = \emptyset$. It follows $C_i \cap C_{i+k} = \emptyset$ for all $k \geq 1$. Therefore, we can define $\gamma_i := \gamma|_{C_i} : C_i \to \gamma(C_i)$ for $i \in \{0, 1, 2, ...\}$ and obtain an infinite sequence of distinct separating elements in Sim.

Observation 3: Assume there is a separating element $\gamma : A \to B$ in Sim with A being an infinite set. Then A has infinitely many subballs and we see at once that there are infinitely many distinct separating elements in Sim.

Observation 4: Assuming we only have finitely many separating elements in Sim, then we claim that there are only finitely many non-identity equalizing elements in Sim. This follows if we show that each $\gamma : C \to C$ in Sim (which is an isometry) is itself locally determined by identities and separating elements. This means that for each $x \in C$ we find a ball $D \subset C$ with $x \in D$ and either $\gamma|_D = \mathrm{id}_D$ or $D \cap \gamma(D) = \emptyset$. We start by noting that for any isometry $\alpha : Y \to Y$ of a compact ultrametric space Y, if $\alpha \neq \mathrm{id}_Y$, then there must be a ball $D \subset Y$ such that $\alpha(D) \cap D = \emptyset$. Now consider the maximal proper subballs of C. Either γ is the identity on such a ball B or γ maps B to another such ball or γ maps B to itself and is not the identity. Only in the last case we have to go a step deeper and consider the maximal proper subballs of B. Since $\gamma|_B \neq \mathrm{id}_B$, we know that there must be a subball $E \subset B$ such that $\gamma(E) \cap E = \emptyset$. Since there are only finitely many separating elements in Sim, we see that this process has to stop at some point. This proves the claim.

i) ⇒ ii): This is clear from the first observation.

ii) ⇒ iii): Sim cannot contain any contracting elements because of the second observation. Because of the fourth observation, there are also only finitely many non-identity equalizing elements. So Sim has only finitely many non-identity elements.

iii) ⇒ i): This is clear from the definition of $\Gamma(\mathrm{Sim})$.

Now we turn to the last statements. The first of these follows from the second observation. From the fourth observation we know that each element in $\Gamma(\mathrm{Sim})$ is locally determined by identities and separating elements in Sim. From the third observation we know that the latter can only be defined on finite subballs (which consist of finitely many isolated points). But we also know that there are only finitely many elements of this type. This proves that $\Gamma(\mathrm{Sim})$ fixes all points of X except possibly a finite subset $Y \subset X$ of isolated points. Since $\Gamma(\mathrm{Sim}|_Y) \cong \Gamma(\mathrm{Sim})$, we can assume without loss of generality that X itself contains only finitely many points. In this case, by the restriction property of a similarity structure, each element in $\Gamma(\mathrm{Sim})$ is locally determined by similarities in Sim of the form $A \to B$ where A and B are singleton balls. The definition

$$x \sim y \quad :\Longleftrightarrow \quad \mathrm{Sim}\big(\{x\}, \{y\}\big) \neq \emptyset$$

gives an equivalence relation on X. Let $X_1, ..., X_n$ be the corresponding equivalence classes. We have

$$\Gamma(\mathrm{Sim}) \cong \Gamma(\mathrm{Sim}|_{X_1}) \times ... \times \Gamma(\mathrm{Sim}|_{X_n})$$

and $\Gamma(\mathrm{Sim}|_{X_i}) \cong \Sigma_{d_i}$ where d_i is the number of elements in X_i. This proves the last claim of the proposition. □

Proposition 2.7. *Let X be a compact ultrametric space and* Sim *a similarity structure on X such that* $\mathrm{Sim}(B_1, B_2) = \emptyset$ *whenever* $\mathrm{depth}(B_1) \neq \mathrm{depth}(B_2)$. *Then the local similarity group* $\Gamma = \Gamma(\mathrm{Sim})$ *is locally finite.*

Proof. First let α be an arbitrary element in Γ. For each $x \in X$ let A_x be the maximal ball with $x \in A_x$ such that there is an element $\alpha_x \in \mathrm{Sim}(A_x, \alpha(A_x))$ and

$\alpha|_{A_x} = \alpha_x$. The set of balls $\{A_x \mid x \in X\}$ is a partition of X called the partition into maximum regions for α. Define

$$\operatorname{depth}(\alpha) := \max\{\operatorname{depth}(A_x) \mid x \in X\}$$

Note that from the assumption on Sim, each similarity in Sim preserves the depth of balls. Let $\alpha, \beta \in \Gamma$ and let $\mathcal{P}$ and $\mathcal{Q}$ be the corresponding partitions into maximum regions. Observe the composition $\beta \circ \alpha$. It is locally determined by Sim on a partition $\mathcal{R}$ of X into balls such that for every $R \in \mathcal{R}$ either $R = P$ for some $P \in \mathcal{P}$ or $R = (\alpha|_P)^{-1}(Q)$ for some $P \in \mathcal{P}$ and some $Q \in \mathcal{Q}$ with $Q \subset \alpha(P)$. It follows

$$\operatorname{depth}(\beta \circ \alpha) \leq \max\{\operatorname{depth}(R) \mid R \in \mathcal{R}\} \leq \max\{\operatorname{depth}(\alpha), \operatorname{depth}(\beta)\}.$$

So if $\alpha_1, ..., \alpha_k \in \Gamma$, we also have

$$\operatorname{depth}(\alpha_1 \circ ... \circ \alpha_k) \leq \max\{\operatorname{depth}(\alpha_1), ..., \operatorname{depth}(\alpha_k)\}. \tag{3}$$

Now let Λ be a subgroup of Γ with finite generating set $\gamma_1, ..., \gamma_n$. From (3) we deduce that

$$\operatorname{depth}(\lambda) \leq \max\{\operatorname{depth}(\gamma_i) \mid i = 1, ..., n\} =: N$$

for each $\lambda \in \Lambda$. We claim that there are only finitely many local similarities γ locally determined by Sim such that $\operatorname{depth}(\gamma) \leq N$. This follows because there are only finitely many balls B with $\operatorname{depth}(B) \leq N$ and each $\operatorname{Sim}(B_1, B_2)$ is finite by definition. $\square$

3. A condition on the similarity structure

Here we introduce a condition on similarity structures used for the proof of Theorem 1.1 and Theorem 1.2.

Definition 3.1. Let X be a compact ultrametric space and Sim a similarity structure on X. We say Sim is *dually contracting* or has a *dual contraction* if there are two disjoint proper subballs B_1 and B_2 of X together with similarities $X \to B_1$ and $X \to B_2$ in Sim.

Remark 3.2. The property in Definition 3.1 is rather a property of the similarity structure than of the local similarity group $\Gamma(\text{Sim})$. To illustrate the precise meaning of this statement, consider the following. Let X be a compact ultrametric space and Sim a similarity structure on it. Remove all elements in Sim of the form $A \to B$ where either $A = X \neq B$ or $A \neq X = B$. Denote the remaining set of similarities by Sim^-. It is easy to see that Sim^- still forms a similarity structure on X. Furthermore, since no similarity in $\text{Sim} \setminus \text{Sim}^-$ can be used to form a local similarity on X, the groups $\Gamma(\text{Sim})$ and $\Gamma(\text{Sim}^-)$ are the same as sets of local similarities on X. However, even if Sim is dually contracting, the similarity structure Sim^- never is. But it can be extended to a dually contracting one. We therefore call a similarity structure *potentially* dually contracting if it can be extended in such a way that the corresponding local similarity groups are the same (as sets of local similarities) and the extension is dually contracting.

Example 3.3. The similarity structures presented in Example 2.2 are dually contracting. So Theorem 1.1 applies to the Nekrashevych-Röver groups $V_d(H)$ and in particular to the Thompson group V.

Example 3.4. If X is a compact ultrametric space and Sim a similarity structure on X such that the local similarity group $\Gamma(\text{Sim})$ is finite, then, by Proposition 2.6, Sim cannot be potentially dually contracting.

Example 3.5. Let X be the end space of the rooted binary tree with the usual order. Let B_1 and B_2 be the two maximal proper subballs of X. Let Sim be the similarity structure generated by the unique order preserving similarity $\gamma : B_1 \to B_2$, i.e. the smallest similarity structure on X containing γ. More precisely, the non-trivial similarities in Sim are the unique order preserving similarites $xA^\omega \to \bar{x}A^\omega$ where $\bar{x}$ is obtained from x by changing the first letter of x, e.g. $\bar{x} = 101$ if $x = 001$. It follows from Proposition 2.6 that $\Gamma = \Gamma(\mathrm{Sim})$ is infinite and from Proposition 2.7 that Γ is locally finite. It is therefore not finitely generated. By the local finiteness it is also elementary amenable and consequently $H_0(\Gamma, \mathcal{N}(\Gamma)) \neq 0$.
However, this similarity structure is not potentially dually contracting. Otherwise there would be a similarity $\delta : X \to A$ with A a proper subball of X. Let C be another proper subball of X with $A \cap C = \emptyset$. We have $C \cap \delta(C) = \emptyset$. The restriction $\hat{\delta} = \delta|_C : C \to \delta(C) =: D$ fits into a local similarity on X. Just define $\alpha : X \to X$ by

$$\alpha|_C := \hat{\delta} \qquad \alpha|_D := \hat{\delta}^{-1} \qquad \alpha|_{X \setminus (C \cup D)} := \mathrm{id}.$$

This local similarity α is not in $\Gamma(\mathrm{Sim})$ because neither $\hat{\delta}$ nor any of its restrictions is an element of Sim.

Remark 3.6. In [6], Farley and Hughes introduced a condition on a similarity structure Sim, called *rich in ball contractions*, which is defined as follows. There exists a constant $c > 0$ such that for every $k \geq c$ and $(B_1, ..., B_k)$ a k-tuple of balls, there is a ball B with at least two maximal proper subballs and an injection

$$\sigma : \{A \mid A \text{ maximal proper subball of } B\} \to \{(B_i, i) \mid 1 \leq i \leq k\}$$

with $[A] = [B_i]$ whenever $\sigma(A) = (B_i, i)$. In *loc. cit.* it is shown that local similarity groups arising from similarity structures having this property and with only finitely many Sim-equivalence classes of balls are of type FP_∞. It is quite clear that rich in ball contractions implies dually contracting, just take $(X, ..., X)$ as a k-tuple of balls.

In the next lemma, we extract the key features of the property dually contracting. Apart from Proposition 3.8, these are the only ones we will need in the proof of the main theorem. So we could have stated them as a definition, but Definition 3.1 is much easier to state and to verify.

Lemma 3.7. *Let X be a compact ultrametric space with dually contracting similarity structure* Sim. *Then there exists a sequence $(S_i)_{i \in \mathbb{N}}$ where each S_i is a set $\{B_i^1, ..., B_i^{n_i}\}$ of pairwise disjoint balls in X satisfying the following two properties.*

i) *For each i, k there exists a similarity $X \to B_i^k$ in* Sim.
ii) $|S_i| \xrightarrow{i \to \infty} \infty$.
iii) *For every $i_0 \in \mathbb{N}$ and every partition $\mathcal{P}$ of X into non-empty closed open subspaces there is an $i \geq i_0$ such that for every $B \in S_i$ there exists $P \in \mathcal{P}$ with $B \subset P$.*

Proof. Let B_1^1 and B_1^2 be two disjoint proper subballs of X and $\gamma_i : X \to B_1^i$ similarities in Sim for $i = 1, 2$. We will define the S_i's inductively. First set $S_1 = \{B_1^1, B_1^2\}$. Now assume $S_i = \{B_i^1, ..., B_i^{n_i}\}$ has been constructed. Then define

$$S_{i+1} := \{\gamma_1(B_i^k), \gamma_2(B_i^k) \mid 1 \leq k \leq n_i\}.$$

It is clear that $|S_i| = 2^i$ so that ii) holds. Using that Sim is closed under restriction and composition, it is easy to show property i). It is also quite clear that the balls in each S_i are pairwise disjoint. For iii), first define

$$\mathrm{depth}(S) = \min\{\mathrm{depth}(B) \mid B \in S\}$$

for any finite set S of balls in X. Then the claim follows from Lemma 2.4 if we show

$$\lim_{i\to\infty} \operatorname{depth}(S_i) = \infty.$$

Note that an application of the contractions γ_1 or γ_2 to a ball increases its depth by at least one. It follows that $\operatorname{depth}(S_i)$ increases by at least one if i increases by one and therefore goes to infinity if i tends to infinity. $\square$

Proposition 3.8. *If* Sim *is a dually contracting similarity structure, then the local similarity group* $\Gamma = \Gamma(\mathrm{Sim})$ *contains a non-abelian free subgroup and is therefore non-amenable.*

Proof. We will identify two elements in Γ, a_1 and a_2, with $\operatorname{ord}(a_1) = 3$ and $\operatorname{ord}(a_2) = 2$. We will also construct disjoint subsets X_1 and X_2 of X such that

$$\begin{aligned} a_1 X_2 &\subset X_1 \\ a_1^2 X_2 &\subset X_1 \\ a_2 X_1 &\subset X_2 \end{aligned}$$

Thus the ping-pong lemma will tell us that the subgroups $H_1 := \langle a_1 \rangle \cong \mathbb{Z}_3$ and $H_2 := \langle a_2 \rangle \cong \mathbb{Z}_2$ together generate a free product in Γ, i.e. $\langle H_1, H_2 \rangle \cong H_1 * H_2 \cong \mathbb{Z}_3 * \mathbb{Z}_2$ is a subgroup of Γ, which itself contains a non-abelian free subgroup. Let's turn to the construction (see Figure 1 below). Let A_1 and A_2 be two disjoint proper subballs of X and $\gamma_i : X \to A_i$ for $i = 1, 2$ two similarities in Sim. Set

$$\begin{aligned} B_1 &:= \gamma_1(A_1) \\ B_2 &:= \gamma_1(A_2) \\ B_3 &:= \gamma_2(A_1) \\ B_4 &:= \gamma_2(A_2) \end{aligned}$$

These are pairwise disjoint balls in X. The similarities γ_1 and γ_2 induce similarities between any pair of the balls A_i and B_i. For example

$$\begin{aligned} \delta_2 &:= \gamma_2|_{A_1} \circ \gamma_1 \circ \gamma_2^{-1} \circ \gamma_1^{-1}|_{B_2} : B_2 \to B_3 \\ \delta_3 &:= \gamma_2|_{A_2} \circ \gamma_2 \circ \gamma_1^{-1} \circ \gamma_2^{-1}|_{B_3} : B_3 \to B_4 \\ \delta_4 &:= \gamma_1|_{A_2} \circ \gamma_2^{-1}|_{B_4} : B_4 \to B_2 \end{aligned}$$

Now define a_1 to be the identity except on the balls B_2, B_3 and B_4 where

$$\begin{aligned} a_1|_{B_2} &:= \delta_2 : B_2 \to B_3 \\ a_1|_{B_3} &:= \delta_3 : B_3 \to B_4 \\ a_1|_{B_4} &:= \delta_4 : B_4 \to B_2 \end{aligned}$$

It is straightforward to verify

$$\begin{aligned} \delta_4 \circ \delta_3 \circ \delta_2 &= \mathrm{id}_{B_2} \\ \delta_2 \circ \delta_4 \circ \delta_3 &= \mathrm{id}_{B_3} \\ \delta_3 \circ \delta_2 \circ \delta_4 &= \mathrm{id}_{B_4} \end{aligned}$$

so that a_1 has order 3. Define a_2 to be the identity except on the balls B_2 and A_2 where

$$\begin{aligned} a_2|_{B_2} &:= \gamma_1^{-1}|_{B_2} : B_2 \to A_2 \\ a_2|_{A_2} &:= \gamma_1|_{A_2} : A_2 \to B_2 \end{aligned}$$

It is trivial to check $a_2^2 = \mathrm{id}_X$. Last but not least define $X_1 := A_2$ and $X_2 = B_2$. It is easy to see from the definitions that the relations at the beginning of the proof hold, so it is completed. $\square$

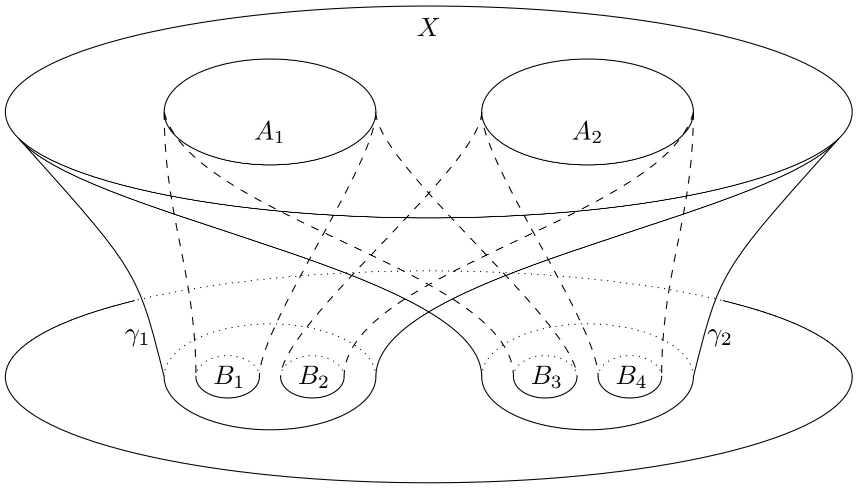

FIGURE 1. Figure for the proof of Proposition 3.8. The ellipses below are copies of the ellipses above and represent the various balls appearing in the proof. The vertical wires represent the similarities γ_i or restrictions of them.

4. Proof of the main theorem

The proof relies on a common feature of relatives of Thompson's groups: they contain products of arbitrarily many copies of themselves as subgroups. This feature was utilized in homological vanishing results before [1, 13].

4.1. **A spectral sequence.** Our main tool will be a spectral sequence explained in Brown's book [2, Chapter VII.7] which we will summarize now. Let Γ be a group and Z a simplicial complex with a simplicial Γ-action. Let M be a $\mathbb{Z}[\Gamma]$-module. For σ a simplex in Z, denote by Γ_σ the isotropy group of σ, i.e. all the elements in Γ which fix σ as a set of vertices. Let M_σ be the orientation $\mathbb{Z}[\Gamma_\sigma]$-module, i.e. $M_\sigma = M$ as an abelian group together with the action

$$\Gamma_\sigma \times M \to M \quad (g, m) \mapsto \begin{cases} gm & \text{if } g \text{ is an even permutation of the vertices of } \sigma \\ -gm & \text{if } g \text{ is an odd permutation of the vertices of } \sigma \end{cases}$$

Furthermore, let Σ_p be a set of p-cells representing the Γ-orbits of Z. Then there is a spectral sequence E^k_{pq} with first term

$$E^1_{pq} = \bigoplus_{\sigma \in \Sigma_p} H_q(\Gamma_\sigma, M_\sigma) \Rightarrow H^\Gamma_{p+q}(Z, M)$$

converging to the Γ-equivariant homology of Z with coefficients in M. In our case, Z will be acyclic, so that $H^\Gamma_{p+q}(Z, M) = H_{p+q}(\Gamma, M)$. Furthermore, Γ_σ will fix σ vertex-wise, so that $M_\sigma = M$ as $\mathbb{Z}[\Gamma_\sigma]$-modules. We therefore have a spectral sequence E^k_{pq} with

$$E^1_{pq} = \bigoplus_{\sigma \in \Sigma_p} H_q(\Gamma_\sigma, M) \Rightarrow H_{p+q}(\Gamma, M).$$

4.2. **The poset of partitions into closed open sets.** Now let $\Gamma = \Gamma(\mathrm{Sim})$ be a local similarity group coming from a dually contracting similarity structure Sim on the compact ultrametric space X. Next we define, for each $n \in \mathbb{N}$, a simplicial Γ-complex Z_n associated to a poset $(\mathbf{P}_n, \leq)$. Unlike the simplicial complex in [6], used for proving finiteness properties, it has large isotropy groups. By definition,

an element in $\mathbf{P}_n$ is a set (*partition*) $\mathcal{P} = \{P_1, ..., P_k\}$ of pairwise disjoint non-empty closed open subspaces of X with $X = P_1 \cup ... \cup P_k$ satisfying the following extra condition: There are at least n elements contained in $\mathcal{P}$ which are locally Sim-equivalent to X.

By Lemma 3.7, $\mathbf{P}_n \neq \emptyset$. Let $\mathcal{P}, \mathcal{Q} \in \mathbf{P}_n$. We say $\mathcal{P} \leq \mathcal{Q}$ iff $\mathcal{Q}$ refines $\mathcal{P}$, that is, $\forall_{Q\in\mathcal{Q}}\exists_{P\in\mathcal{P}}\ Q \subset P$. Then $(\mathbf{P}_n, \leq)$ is a poset. Explicitly, a simplex in the associated simplicial complex Z_n is a finite set of vertices which can be totally ordered using the partial order on $\mathbf{P}_n$. We write $\{\mathcal{P}_1 < ... < \mathcal{P}_k\}$ for such a $(k-1)$-simplex.

Next we will show that $(\mathbf{P}_n, \leq)$ is directed, which implies that Z_n is contractible. So let $\mathcal{P}, \mathcal{Q} \in \mathbf{P}_n$. First, it is easy to see that there is a partition $\mathcal{R}$ into open and closed sets which refines both $\mathcal{P}$ and $\mathcal{Q}$. But we have to refine it even more so that it satisfies the extra condition. From Lemma 3.7 part iii) and ii) we obtain that there are at least n disjoint balls $B_1, ..., B_s$ such that every ball B_i is contained in some element of $\mathcal{R}$ and from part i) we know that every ball B_i is Sim-equivalent to X. So we can take these balls as elements of a refinement of $\mathcal{R}$.

We endow Z_n with the simplicial Γ-action

$$g\{P_1, ..., P_k\} = \{g(P_1), ..., g(P_k)\}$$

for a vertex $\mathcal{P} = \{P_1, ..., P_k\}$ and $g \in \Gamma$. We have $g\mathcal{P} \leq g\mathcal{Q}$ whenever $\mathcal{P} \leq \mathcal{Q}$. It follows that the action is indeed simplicial and that whenever $g \in \Gamma$ fixes a simplex as a set of vertices, then it fixes it vertex-wise.

We want to take a closer look at the isotropy groups Γ_σ for $\sigma = \{\mathcal{P}_1 < ... < \mathcal{P}_k\}$ a simplex. First consider the case $k = 1$. Write $\mathcal{P} = \mathcal{P}_1$. If $g \in \Gamma_\sigma$ then $g\mathcal{P} = \mathcal{P}$ and consequently, there is a permutation π of the set $\mathcal{P}$ such that $g(P) = \pi(P)$ for every $P \in \mathcal{P}$. Write $\Sigma_\mathcal{P}$ for the group of permutations of the set $\mathcal{P}$. We therefore have

$$\Gamma_\sigma = \bigl\{g \in \Gamma \mid \exists_{\pi\in\Sigma_\mathcal{P}}\forall_{P\in\mathcal{P}}\ g(P) = \pi(P)\bigr\}.$$

Now let $k \geq 1$ be arbitrary. In this case, we have $g\mathcal{P}_i = \mathcal{P}_i$ for each $i = 1, ..., k$. First we start with a preliminary remark. Let $\mathcal{P} \leq \mathcal{Q}$ be two vertices. Then there is a unique function $f : \mathcal{Q} \to \mathcal{P}$ such that $Q \subset f(Q)$ for all $Q \in \mathcal{Q}$. Let $\pi \in \Sigma_\mathcal{Q}$. Then π is called $\mathcal{P}$-admissible iff there is a $\rho \in \Sigma_\mathcal{P}$ such that for all $P \in \mathcal{P}$ and all $Q \in f^{-1}(P)$ we have $f(\pi(Q)) = \rho(P)$. In other words, π is a permutation of $\mathcal{Q}$ which gives a permutation of $\mathcal{P}$ when we write each element in $\mathcal{P}$ as a disjoint union of elements in $\mathcal{Q}$. The set of all $\mathcal{P}$-admissible elements forms a subgroup of $\Sigma_\mathcal{Q}$, denoted by $\Sigma_{\mathcal{P}\leq\mathcal{Q}}$. More generally, if we have an ascending chain $\mathcal{Q}_1 \leq ... \leq \mathcal{Q}_l$ of vertices, then we can define the subgroup $\Sigma_{\mathcal{Q}_1\leq...\leq\mathcal{Q}_l}$ of $\Sigma_{\mathcal{Q}_l}$ consisting of all elements in $\Sigma_{\mathcal{Q}_l}$ which are $\mathcal{Q}_i$-admissible for all $i = 1, ..., l-1$. In particular, we have defined a subgroup Σ_σ of $\Sigma_{\mathcal{P}_k}$ for the simplex $\sigma = \{\mathcal{P}_1 < ... < \mathcal{P}_k\}$ from above. This group is defined exactly in a way such that

$$\Gamma_\sigma = \bigl\{g \in \Gamma \mid \exists_{\pi\in\Sigma_\sigma}\forall_{P\in\mathcal{P}_k}\ g(P) = \pi(P)\bigr\}.$$

The group

$$\Lambda_\sigma = \bigl\{g \in \Gamma \mid \forall_{P\in\mathcal{P}_k}\ g(P) = P\bigr\} \cong \prod_{P\in\mathcal{P}_k} \Gamma(\mathrm{Sim}|_P)$$

is a normal subgroup of Γ_σ. It is also of finite index in Γ_σ because the quotient $\Gamma_\sigma/\Lambda_\sigma$ injects into the finite group Σ_σ.

4.3. **Künneth theorems.** The following Künneth vanishing result was proved in the context of Farber's extended l^2-homology in [5, 3. Appendix]. For the proof of the exact formulation below see [12, Lemma 12.11 on p. 448].

Proposition 4.1. *Let $G = G_1 \times G_2$ be a product of two groups. Assume that $H_p(G_1, \mathcal{N}(G_1)) = 0$ for $p \leq n_1$ and $H_p(G_2, \mathcal{N}(G_2)) = 0$ for $p \leq n_2$. Then we have*

$H_p(G, \mathcal{N}(G)) = 0$ for $p \leq n_1 + n_2 + 1$. Note that the case $n_i = -1$ is allowed and gives a non-trivial statement.

Corollary 4.2. *Let $m \geq n \geq 2$ and $G = G_1 \times ... \times G_m$ be a product of m groups. Assume that $H_0(\bullet, \mathcal{N}(\bullet))$ vanishes for at least n of the groups G_i. Then $H_*(G, \mathcal{N}(G))$ vanishes up to degree $n-1$.*

We also need cohomological versions of these results with coefficients in the group ring.

Proposition 4.3. *Let $G = G_1 \times G_2$ be a product of two groups of type FP_∞. Assume that $H^p(G_1, \mathbb{Z}[G_1]) = 0$ for $p \leq n_1$ and $H^p(G_2, \mathbb{Z}[G_2]) = 0$ for $p \leq n_2$. Then we have $H^p(G, \mathbb{Z}[G]) = 0$ for $p \leq n_1 + n_2 + 1$.*

Proof. Let P_* be a $\mathbb{Z}[G_1]$-resolution of $\mathbb{Z}$ such that each P_i is a finitely generated free $\mathbb{Z}[G_1]$-module. Let Q_* be a similar resolution for G_2. Then

$$C^* = \hom_{\mathbb{Z}[G_1]}(P_*, \mathbb{Z}[G_1]) \quad \text{and} \quad D^* = \hom_{\mathbb{Z}[G_2]}(Q_*, \mathbb{Z}[G_2])$$

are cochain complexes of free abelian groups which compute $H^*(G_1, \mathbb{Z}[G_1])$ and $H^*(G_2, \mathbb{Z}[G_2])$ respectively. For $\mathbb{Z}[G_i]$-modules M_i, $i \in \{1, 2\}$, the cochain cross product [2, Chapter V.2]

$$\hom_{\mathbb{Z}[G_1]}(P_*, M_1) \otimes_{\mathbb{Z}} \hom_{\mathbb{Z}[G_2]}(Q_*, M_2) \to \hom_{\mathbb{Z}[G]}(P_* \otimes_{\mathbb{Z}} Q_*, M_1 \otimes_{\mathbb{Z}} M_2) \tag{4}$$

is an isomorphism of cochain complexes since all P_i and Q_j are finitely generated free. If $M_i = \mathbb{Z}[G_i]$, then $M_1 \otimes_{\mathbb{Z}} M_2 \cong \mathbb{Z}[G]$ as $\mathbb{Z}[G]$-modules and the right hand side of (4) computes $H^*(G, \mathbb{Z}[G])$ [2, Proposition (1.1) on p. 107]. By a suitable Künneth theorem [4, Theorem 9.13 on p. 164] and the fact that C^*, D^* are free as $\mathbb{Z}$-modules, the homology of $C^* \otimes_{\mathbb{Z}} D^*$ vanishes in degrees $\leq n_1 + n_2 + 1$. □

The following corollary follows from $H^0(G, \mathbb{Z}[G]) \cong \mathbb{Z}[G]^G = 0$ for infinite G.

Corollary 4.4. *Let $G = G_1 \times ... \times G_n$ be a product of n infinite FP_∞ groups. Then $H^*(G, \mathbb{Z}[G])$ vanishes up to degree $n-1$.*

4.4. **Proof of Theorem 1.1.** Let $n \geq 2$ be arbitrary. Consider the simplicial Γ-complex Z_n from above. From the discussion it follows that we have a spectral sequence E^k_{pq} with

$$E^1_{pq} = \bigoplus_{\sigma \in \Sigma_p} H_q(\Gamma_\sigma, \mathcal{N}(\Gamma)) \Rightarrow H_{p+q}(\Gamma, \mathcal{N}(\Gamma)). \tag{5}$$

Fix a simplex $\sigma = \{\mathcal{P}_1 < ... < \mathcal{P}_p\}$. First observe the group $\Lambda_\sigma \cong \prod_{P \in \mathcal{P}_p} \Gamma(\mathrm{Sim}|_P)$ defined above. From the extra condition on the vertices we know that at least n elements of $\mathcal{P}_p$ are locally Sim-equivalent to X and therefore, by Lemma 2.3, at least n of the groups $\Gamma(\mathrm{Sim}|_P)$ with $P \in \mathcal{P}_p$ are isomorphic to Γ. Γ is infinite by Proposition 2.6 and non-amenable by Proposition 3.8. Going back to a result by Kesten [12, Lemma 6.36 on p. 258], this is equivalent to $H_0(\Gamma, \mathcal{N}(\Gamma)) = 0$. By Corollary 4.2 we therefore have $H_q(\Lambda_\sigma, \mathcal{N}(\Lambda_\sigma)) = 0$ for $q = 0, ..., n-1$. Since Λ_σ is normal in Γ_σ we have $H_q(\Gamma_\sigma, \mathcal{N}(\Gamma_\sigma)) = 0$ for $q = 0, ..., n-1$ by [12, Lemma 12.11]. Since $\mathcal{N}(\Gamma)$ is a flat ring extension of $\mathcal{N}(\Gamma_\sigma)$ [12, Theorem 6.29], it follows that

$$H_q(\Gamma_\sigma, \mathcal{N}(\Gamma)) = 0 \quad \text{for } q \in \{0, ..., n-1\}.$$

Consequently, the spectral sequence (5) collapses except possibly in the region $p \geq 0$, $q \geq n$ and therefore

$$H_i(\Gamma, \mathcal{N}(\Gamma)) = 0 \quad \text{for } i \leq n-1.$$

Because n is arbitrary, Theorem 1.1 follows.

4.5. **Proof of Theorem 1.2.** The proof is similar to the one above and we only describe the necessary modifications. Hughes and Farley proved that Γ is of type FP_∞ under the assumptions on Sim [6, Theorem 1.1] and it is infinite because of Proposition 2.6. Instead of (5), we use the cohomological version of Brown's spectral sequence with coefficients in the group ring:

$$E_1^{pq} = \prod_{\sigma\in\Sigma_p} H^q(\Gamma_\sigma, \mathbb{Z}[\Gamma]) \Rightarrow H^{p+q}(\Gamma, \mathbb{Z}[\Gamma]).$$

Write $\mathcal{P}_p = \{P_1, ..., P_k\}$ such that the first n elements are locally Sim-equivalent to X. Observe the normal subgroup

$$\Lambda'_\sigma = \prod_{i=1}^n \Gamma(\mathrm{Sim}|_{P_i}) \lhd \Lambda_\sigma = \prod_{i=1}^k \Gamma(\mathrm{Sim}|_{P_i}).$$

By Corollary 4.4 we obtain $H^q(\Lambda'_\sigma, \mathbb{Z}[\Lambda'_\sigma]) = 0$ for $q \in \{0, \dots, n-1\}$. Since $\mathbb{Z}[\Gamma]$ is a free $\mathbb{Z}[\Lambda'_\sigma]$-module and group cohomology of FP_∞-groups commutes with direct limits in the coefficients [2, Theorem (4.8) on p. 196], we obtain $H^q\big(\Lambda'_\sigma, \mathbb{Z}[\Gamma]\big) = 0$ for $q \in \{0, \dots, n-1\}$. Now an application of the cohomological Hochschild-Lyndon-Serre spectral sequence to the group extension

$$1 \to \Lambda'_\sigma \to \Lambda_\sigma \to \Lambda_\sigma/\Lambda'_\sigma \to 1$$

and the coefficient module $\mathbb{Z}[\Gamma]$ yields $H^q(\Lambda_\sigma, \mathbb{Z}[\Gamma]) = 0$ for $q \in \{0, \dots, n-1\}$. Apply this spectral sequence once more to the group extension

$$1 \to \Lambda_\sigma \to \Gamma_\sigma \to \Gamma_\sigma/\Lambda_\sigma \to 1$$

to obtain

$$H^q(\Gamma_\sigma, \mathbb{Z}[\Gamma]) = 0 \quad \text{for } q \in \{0, \dots, n-1\}.$$

Now proceed as above.

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